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# Sharp estimates for the convergence of the density of the Euler scheme in small time

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## Abstract

In this work, we approximate a diffusion process by its Euler scheme and we study the convergence of the density of the marginal laws. We improve previous estimates especially for small time.

**Keywords :** stochastic differential equation, Euler scheme, rate of convergence, Malliavin calculus.

AMS classification: 65C20 60H07 60H10 65G99 65M15 60J60

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# 1 Introduction

Let us consider a  $d$ -dimensional diffusion process  $(X_s)_{0 \leq s \leq T}$  and a  $q$ -dimensional Brownian motion  $(W_s)_{0 \leq s \leq T}$ .  $X$  satisfies the following SDE

$$dX_s^i = b_i(s, X_s)ds + \sum_{j=1}^q \sigma_{ij}(s, X_s)dW_s^j, \quad X_0^i = x^i, \forall i \in \{1, \dots, d\}. \quad (1.1)$$

We approximate  $X$  by its Euler scheme with  $N$  ( $N \geq 1$ ) time steps, say  $X^N$ , defined as follows. We consider the regular grid  $\{0 = t_0 < t_1 < \dots < t_N = T\}$  of the interval  $[0, T]$ , i.e.  $t_k = k\frac{T}{N}$ . We put  $X_0^N = x$  and for all  $i \in \{1, \dots, d\}$  we define

$$X_u^{N,i} = X_{t_k}^{N,i} + b_i(t_k, X_{t_k}^N)(u - t_k^N) + \sum_{j=1}^q \sigma_{ij}(t_k, X_{t_k}^N)(W_u^j - W_{t_k}^j), \text{ for } u \in [t_k, t_{k+1}[. \quad (1.2)$$

The continuous Euler scheme is an Itô process verifying

$$X_u^N = x + \int_0^u b(\varphi(s), X_{\varphi(s)}^N)ds + \int_0^u \sigma(\varphi(s), X_{\varphi(s)}^N)dW_s$$

where  $\varphi(u) := \sup\{t_k : t_k \leq u\}$ . If  $\sigma$  is uniformly elliptic, the Markov process  $X$  admits a transition probability density  $p(0, x; s, y)$ . Concerning  $X^N$  (which is not Markovian except at times  $(t_k)_k$ ),  $X_s^N$  has a probability density  $p^N(0, x; s, y)$ , for any  $s > 0$ . We aim at proving sharp estimates of the difference  $p(0, x; s, y) - p^N(0, x; s, y)$ .

It is well known (see Bally and Talay (1996), Konakov and Mammen (2002), Guyon (2006)) that this difference is of order  $\frac{1}{N}$ . However, the known upper bounds of this difference are too rough for small values of  $s$ . In this work, we provide tight upper bounds of  $|p(0, x; s, y) - p^N(0, x; s, y)|$  in  $s$  (see Theorem 3), so that we can estimate quantities like

$$\mathbb{E}[f(X_T^N)] - \mathbb{E}[f(X_T)] \text{ or } \mathbb{E}\left[\int_0^T f(X_{\varphi(s)}^N)ds\right] - \mathbb{E}\left[\int_0^T f(X_s)ds\right] \quad (1.3)$$

(without any regularity assumptions on  $f$ ) more accurately than before (see Theorem 5). For other applications, see Labart (2007). Unlike previous references, we allow  $b$  and  $\sigma$  to be time-dependent and assume they are only  $C^3$  in space. Besides, we use Malliavin's calculus tools.

## Background results

The difference  $p(0, x; s, y) - p^N(0, x; s, y)$  has been studied a lot. We can found several results in the literature on expansions w.r.t.  $N$ . First, we mention a result from Bally and Talay (1996) (Corollary 2.7). The authors assume

**Hypothesis 1**  $\sigma$  is elliptic (with  $\sigma$  only depending on  $x$ ) and  $b, \sigma$  are  $C^\infty(\mathbb{R}^d)$  functions whose derivatives of any order greater or equal to 1 are bounded.

By using Malliavin's calculus, they show that

$$p(0, x; T, y) - p^N(0, x; T, y) = \frac{1}{N}\pi_T(x, y) + \frac{1}{N^2}R_T^N(x, y), \quad (1.4)$$

with  $|\pi_T(x, y)| + |R_T^N(x, y)| \leq \frac{K(T)}{T^q} \exp(-c \frac{|x-y|^2}{T})$ , where  $c > 0$ ,  $q > 0$  and  $K(\cdot)$  is a non decreasing function. We point out that  $q$  is unknown, which doesn't enable to deduce the behavior of  $p - p^N$  when  $T \rightarrow 0$ .

Besides that, Konakov and Mammen (2002) have proposed an analytical approach based on the so-called parametrix method to bound  $p(0, x; 1, y) - p^N(0, x; 1, y)$  from above. They assume

**Hypothesis 2**  $\sigma$  is elliptic and  $b, \sigma$  are  $C^\infty(\mathbb{R}^d)$  functions whose derivatives of any order are bounded.

For each pair  $(x, y)$  they get an expansion of arbitrary order  $j$  of  $p^N(0, x; 1, y)$ . The coefficients of the expansion depend on  $N$

$$p(0, x; 1, y) - p^N(0, x; 1, y) = \sum_{i=1}^{j-1} \frac{1}{N^i} \pi_{N,i}(0, x; 1, y) + O\left(\frac{1}{N^j}\right). \quad (1.5)$$

The coefficients have Gaussian tails : for each  $i$  they find constants  $c_1 > 0$ ,  $c_2 > 0$  s.t. for all  $N \geq 1$  and all  $x, y \in \mathbb{R}^d$ ,  $|\pi_{N,i}(0, x; 1, y)| \leq c_1 \exp(-c_2|x - y|^2)$ . To do so, they use upper bounds for the partial derivatives of  $p$  (coming from Friedman (1964)) and prove analogous results on the derivatives of  $p^N$ . Strong though this result may be, nothing is said when replacing 1 by  $t$ , for  $t \rightarrow 0$ . That's why we present now the work of Guyon (2006).

Guyon (2006) improves (1.4) and (1.5) in the following way.

**Definition 1.** Let  $\mathcal{G}_l(\mathbb{R}^d)$ ,  $l \in \mathbb{Z}$  be the set of all measurable functions  $\pi : \mathbb{R}^d \times (0, 1] \times \mathbb{R}^d \rightarrow \mathbb{R}$  s.t.

- for all  $t \in (0, 1]$ ,  $\pi(\cdot; t, \cdot)$  is infinitely differentiable,
- for all  $\alpha, \beta \in \mathbb{N}^d$ , there exist two constants  $c_1 \geq 0$  and  $c_2 > 0$  s.t. for all  $t \in (0, 1]$  and  $x, y \in \mathbb{R}^d$ ,

$$|\partial_x^\alpha \partial_y^\beta \pi(x; t, y)| \leq c_1 t^{-(|\alpha|+|\beta|+d+l)/2} \exp(-c_2|x - y|^2/t).$$

Under Hypothesis 2 and for  $T = 1$ , the author has proved the following expansions

$$p^N - p = \frac{\pi}{N} + \frac{\pi_N}{N^2}, \quad (1.6)$$

$$p^N - p = \sum_{i=1}^{j-1} \frac{\pi_{N,i}}{N^i} + \sum_{i=2}^j \left( t - \frac{\lfloor Nt \rfloor}{N} \right)^i \pi'_{N,i} + \frac{\pi''_{N,j}}{N^j}, \quad (1.7)$$

where  $\pi \in \mathcal{G}_1(\mathbb{R}^d)$  and  $(\pi_N, N \geq 1)$  is a bounded sequence in  $\mathcal{G}_4(\mathbb{R}^d)$ . For each  $i \geq 1$ ,  $(\pi_{N,i}, N \geq 1)$  is a bounded family in  $\mathcal{G}_{2i-2}(\mathbb{R}^d)$ , and  $(\pi'_{N,i}, N \geq 1)$ ,  $(\pi''_{N,i}, N \geq 1)$  are two bounded families in  $\mathcal{G}_{2i}(\mathbb{R}^d)$ . These expansions can be seen as improvements of (1.4) and (1.5) : it also allows infinite differentiations w.r.t.  $x$  and  $y$  and makes precise the way the coefficients explode when  $t$  tends to 0.

As a consequence (see Guyon (2006, Corollary 22)), one gets

$$|p(0, x; s, y) - p^N(0, x; s, y)| \leq \frac{c_1}{Ns^{\frac{d+2}{2}}} e^{-c_2 \frac{|x-y|^2}{s}}, \quad (1.8)$$

for two positive constants  $c_1$  and  $c_2$ , and for any  $x, y$  and  $s \leq 1$ . This result should be compared with the one of Theorem 3 (when  $T = 1$ ), in which the upper bound is tighter ( $s$  has a smaller power).

## 2 Main Results

Before stating the main result of the paper, we introduce the following notation

**Definition 2.**  $C_b^{k,l}$  denotes the set of continuously differentiable bounded functions  $\phi : (t, x) \in [0, T] \times \mathbb{R}^d$  with uniformly bounded derivatives w.r.t.  $t$  (resp. w.r.t.  $x$ ) up to order  $k$  (resp. up to order  $l$ ).

The main result of the paper, whose proof is postponed to Section 4, is established under the following Hypothesis

**Hypothesis 3**  $\sigma$  is uniformly elliptic,  $b$  and  $\sigma$  are in  $C_b^{1,3}$  and  $\partial_t \sigma$  is in  $C_b^{0,1}$ .

**Theorem 3.** Assume Hypothesis 3. Then, there exist a constant  $c > 0$  and a non decreasing function  $K$ , depending on the dimension  $d$  and on the upper bounds of  $\sigma, b$  and their derivatives s.t.  $\forall (s, x, y) \in ]0, T] \times \mathbb{R}^d \times \mathbb{R}^d$ , one has

$$|p(0, x; s, y) - p^N(0, x; s, y)| \leq \frac{K(T)T}{Ns^{\frac{d+1}{2}}} \exp\left(-\frac{c|x-y|^2}{s}\right).$$

**Corollary 4.** Assume Hypothesis 3. From the last inequality and Aronson's inequality (A.1), we deduce

$$\left| \frac{p(0, x; T, x) - p^N(0, x; T, x)}{p(0, x; T, x)} \right| \leq \frac{K(T)}{N} \sqrt{T}. \quad (2.1)$$

This inequality yields  $p(0, x; T, x) \sim p^N(0, x; T, x)$  when  $T \rightarrow 0$ .

Theorem 3 enables to bound quantities like in (1.3) in the following way

**Theorem 5.** Assume Hypothesis 3. For any function  $f$  such that  $|f(x)| \leq c_1 e^{c_2|x|}$ , it holds

$$\begin{aligned} |\mathbb{E}[f(X_T^N)] - \mathbb{E}[f(X_T)]| &\leq c_1 e^{c_2|x|} K(T) \frac{\sqrt{T}}{N}, \\ \left| \mathbb{E} \left[ \int_0^T f(X_{\varphi(s)}^N) ds \right] - \mathbb{E} \left[ \int_0^T f(X_s) ds \right] \right| &\leq c_1 e^{c_2|x|} K(T) \frac{T}{N}. \end{aligned}$$

Had we used the results stated by Guyon (2006) (and more precisely the one recalled in (1.8)), we would have obtained  $\mathbb{E}[f(X_T^N)] - \mathbb{E}[f(X_T)] = O(\frac{1}{N})$ . Intuitively, this result is not optimal: the right hand side doesn't tend to 0 when  $T$  goes to 0 while it should. Analogously, regarding  $\mathbb{E} \left[ \int_0^T f(X_{\varphi(s)}^N) ds \right] - \mathbb{E} \left[ \int_0^T f(X_{\varphi(s)}) ds \right]$ , we would obtain  $O(\frac{T \ln N}{N})$  instead of  $O(\frac{T}{N})$ .

*Proof of Theorem 5.* Writing  $\mathbb{E}[f(X_T^N)] - \mathbb{E}[f(X_T)]$  as  $\int_{\mathbb{R}^d} f(y)(p^N(0, x; T, y) - p(0, x; T, y))dy$  and using Theorem 3 yield the first result. Concerning the second result, we split  $\mathbb{E}\left[\int_0^T (f(X_{\varphi(s)}^N) - f(X_s))ds\right]$  in two terms  $\mathbb{E}\left[\int_0^T (f(X_{\varphi(s)}^N) - f(X_{\varphi(s)}))ds\right]$  and  $\mathbb{E}\left[\int_0^T (f(X_{\varphi(s)}) - f(X_s))ds\right]$ . First, using Theorem 3 leads to

$$\begin{aligned} \left| \mathbb{E}\left[\int_0^T (f(X_{\varphi(s)}^N) - f(X_{\varphi(s)}))ds\right] \right| &= \left| \int_{\mathbb{R}^d} dy \int_{\frac{T}{N}}^T ds f(y)(p^N(0, x; \varphi(s), y) - p(0, x; \varphi(s), y)) \right|, \\ &\leq \frac{K(T)T}{N} c_1 e^{c_2|x|} \int_{\frac{T}{N}}^T \frac{ds}{\sqrt{\varphi(s)}}, \end{aligned}$$

where we use the easy inequality  $\int_{\mathbb{R}^d} e^{c_2|y|} \frac{e^{-\frac{c|x-y|^2}{s}}}{s^{d/2}} dy \leq K(T)e^{c_2|x|}$ . Since  $\varphi(s) \geq s - \frac{T}{N}$ , we get  $\left| \mathbb{E}\left[\int_0^T (f(X_{\varphi(s)}^N) - f(X_{\varphi(s)}))ds\right] \right| \leq \frac{K(T)T^{3/2}}{N} c_1 e^{c_2|x|}$ . Second, we write

$$\left| \mathbb{E}\left[\int_0^T (f(X_{\varphi(s)}) - f(X_s))ds\right] \right| \leq c_1 e^{c_2|x|} \frac{T}{N} + \int_{\mathbb{R}^d} dy \int_{\frac{T}{N}}^T ds c_1 e^{c_2|y|} \int_{\varphi(s)}^s du |\partial_u p(0, x; u, y)|.$$

Then, Proposition 13 yields  $\left| \mathbb{E}\left[\int_0^T (f(X_{\varphi(s)}) - f(X_s))ds\right] \right| \leq c_1 e^{c_2|x|} \left( \frac{T}{N} + C \int_{\frac{T}{N}}^T \ln\left(\frac{s}{\varphi(s)}\right) ds \right)$ . Moreover,  $\int_{\frac{T}{N}}^T \ln\left(\frac{s}{\varphi(s)}\right) ds = \sum_{k=1}^{N-1} \int_{t_k}^{t_{k+1}} \ln\left(\frac{s}{t_k}\right) ds = \frac{T}{N} \sum_{k=1}^{N-1} ((k+1) \ln\left(\frac{k+1}{k}\right) - 1) \leq C \frac{T}{N}$ , using a second order Taylor expansion. This gives  $\left| \mathbb{E}\left[\int_0^T (f(X_{\varphi(s)}) - f(X_s))ds\right] \right| \leq c_1 e^{c_2|x|} K(T) \frac{T}{N}$ .  $\blacksquare$

In the next section, we give results related to Malliavin's calculus, that will be useful for the proof of Theorem 3.

### 3 Basic results on Malliavin's calculus

We refer the reader to Nualart (2006), for more details. Fix a filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$  and let  $(W_t)_{t \geq 0}$  be a  $q$ -dimensional Brownian motion. For  $h(\cdot) \in H = \mathbb{L}^2([0, T], \mathbb{R}^q)$ ,  $W(h)$  is the Wiener stochastic integral  $\int_0^T h(t) dW_t$ . Let  $\mathcal{S}$  denote the class of random variables of the form  $F = f(W(h_1), \dots, W(h_n))$  where  $f$  is a  $C^\infty$  function with derivatives having a polynomial growth,  $(h_1, \dots, h_n) \in H^n$  and  $n \geq 1$ . For  $F \in \mathcal{S}$ , we define its derivative  $\mathcal{D}F = (\mathcal{D}_t F := (\mathcal{D}_t^1 F, \dots, \mathcal{D}_t^q F))_{t \in [0, T]}$  as the  $H$  valued random variable given by

$$\mathcal{D}_t F = \sum_{i=1}^n \partial_{x_i} f(W(h_1), \dots, W(h_n)) h_i(t).$$

The operator  $\mathcal{D}$  is closable as an operator from  $\mathbb{L}^p(\Omega)$  to  $\mathbb{L}^p(\Omega; H)$ , for  $p \geq 1$ . Its domain is denoted by  $\mathbb{D}^{1,p}$  w.r.t. the norm  $\|F\|_{1,p} = [\mathbb{E}|F|^p + \mathbb{E}(\|\mathcal{D}F\|_H^p)]^{1/p}$ . We can define the iteration of the operator  $\mathcal{D}$ , in such a way that for a smooth random variable  $F$ , the derivative  $\mathcal{D}^k F$  is a random variable with values on  $H^{\otimes k}$ . As in the case  $k = 1$ , the operator  $\mathcal{D}^k$  is closable from  $\mathcal{S} \subset \mathbb{L}^p(\Omega)$  into  $\mathbb{L}^p(\Omega; H^{\otimes k})$ ,  $p \geq 1$ . If we define the norm

$$\|F\|_{k,p} = [\mathbb{E}|F|^p + \sum_{j=1}^k \mathbb{E}(\|\mathcal{D}^j F\|_{H^{\otimes j}}^p)]^{1/p},$$

we denote its domain by  $\mathbb{D}^{k,p}$ . Finally, set  $\mathbb{D}^{k,\infty} = \cap_{p \geq 1} \mathbb{D}^{k,p}$ , and  $\mathbb{D}^\infty = \cap_{k,p \geq 1} \mathbb{D}^{k,p}$ . One has the following chain rule property

**Proposition 6.** *Fix  $p \geq 1$ . For  $f \in C_b^1(\mathbb{R}^d, \mathbb{R})$ , and  $F = (F_1, \dots, F_d)^*$  a random vector whose components belong to  $\mathbb{D}^{1,p}$ ,  $f(F) \in \mathbb{D}^{1,p}$  and for  $t \geq 0$ , one has  $\mathcal{D}_t(f(F)) = f'(F)\mathcal{D}_tF$ , with the notation*

$$\mathcal{D}_tF = \begin{pmatrix} \mathcal{D}_tF_1 \\ \vdots \\ \mathcal{D}_tF_d \end{pmatrix} \in \mathbb{R}^d \otimes \mathbb{R}^q.$$

We now introduce the Skorohod integral  $\delta$ , defined as the adjoint operator of  $\mathcal{D}$ .

**Proposition 7.**  *$\delta$  is a linear operator on  $\mathbb{L}^2([0, T] \times \Omega, \mathbb{R}^q)$  with values in  $\mathbb{L}^2(\Omega)$  s.t.*

- *the domain of  $\delta$  (denoted by  $\text{Dom}(\delta)$ ) is the set of processes  $u \in \mathbb{L}^2([0, T] \times \Omega, \mathbb{R}^q)$  s.t.  $|\mathbb{E}(\int_0^T \mathcal{D}_tF \cdot u_t dt)| \leq c(u)|F|_{\mathbb{L}^2}$  for any  $F \in \mathbb{D}^{1,2}$ .*
- *If  $u$  belongs to  $\text{Dom}(\delta)$ , then  $\delta(u)$  is the one element of  $\mathbb{L}^2(\Omega)$  characterized by the integration by parts formula*

$$\forall F \in \mathbb{D}^{1,2}, \quad \mathbb{E}(F\delta(u)) = \mathbb{E}\left(\int_0^T \mathcal{D}_tF \cdot u_t dt\right).$$

**Remark 8.** *If  $u$  is an adapted process belonging to  $\mathbb{L}^2([0, T] \times \Omega, \mathbb{R}^q)$ , then the Skorohod integral and the Itô integral coincide :  $\delta(u) = \int_0^T u_t dW_t$ , and the preceding integration by parts formula becomes*

$$\forall F \in \mathbb{D}^{1,2}, \quad \mathbb{E}\left(F \int_0^T u_t dW_t\right) = \mathbb{E}\left(\int_0^T \mathcal{D}_tF \cdot u_t dt\right). \quad (3.1)$$

*This equality is also called the duality formula.*

This duality formula is the corner stone to establish general integration by parts formula of the form

$$\mathbb{E}[\partial^\alpha g(F)G] = \mathbb{E}[g(F)H_\alpha(F, G)]$$

for any non degenerate random variables  $F$ . We only give the formulation in the case of interest  $F = X_t^N$ .

**Proposition 9.** *We assume that  $\sigma$  is uniformly elliptic and  $b$  and  $\sigma$  are in  $C_b^{0,3}$ . For all  $p > 1$ , for all multi-index  $\alpha$  s.t.  $|\alpha| \leq 2$ , for all  $t \in ]0, T]$ , all  $u, r, s \in [0, T]$  and for any functions  $f$  and  $g$  in  $C_b^{|\alpha|}$ , there exist a random variable  $H_\alpha \in \mathbb{L}^p$  and a function  $K(T)$  (uniform in  $N, x, s, u, r, t, f$  and  $g$ ) s.t.*

$$\mathbb{E}[\partial_x^\alpha f(X_t^N)g(X_u^N, X_r^N, X_s^N)] = \mathbb{E}[f(X_t^N)H_\alpha], \quad (3.2)$$

*with*

$$|H_\alpha|_{\mathbb{L}^p} \leq \frac{K(T)}{t^{\frac{|\alpha|}{2}}} \|g\|_{C_b^{|\alpha|}}. \quad (3.3)$$

These results are given in the article of Kusuoka and Stroock (1984): (3.3) is owed to Theorem 1.20 and Corollary 3.7.

Another consequence of the duality formula is the derivation of an upper bound for  $p^N$ .

**Proposition 10.** *Assume  $\sigma$  is uniformly elliptic and  $b$  and  $\sigma$  are in  $C_b^{0,2}$ . Then, for any  $x, y \in \mathbb{R}^d$ ,  $s \in ]0, T]$ , one has*

$$p^N(0, x; s, y) \leq \frac{K(T)}{s^{d/2}} e^{-c \frac{|x-y|^2}{s}}, \quad (3.4)$$

for a positive constant  $c$  and a non decreasing function  $K$ , both depending on  $d$  and on the upper bounds for  $b, \sigma$  and their derivatives.

Although this upper bound seems to be quite standard, to our knowledge such a result has not appeared in the literature before, except in the case of time homogeneous coefficients (see Konakov and Mammen (2002), proof of Theorem 1.1).

*Proof.* The inequality (1.32) of Kusuoka and Stroock (1984, Theorem 1.31) gives  $p^N(0, x; s, y) \leq \frac{K(T)}{s^{d/2}}$  for any  $x$  and  $y$ . This implies the required upper bound when  $|x - y| \leq \sqrt{s}$ . Let us now consider the case  $|x - y| > \sqrt{s}$ . Using the same notations as in Kusuoka and Stroock (1984), we denote  $\psi(y) = \rho(\frac{|y-x|}{r})$  where  $r > 0$  and  $\rho$  is a  $C_b^\infty$  function such that  $\mathbf{1}_{\{3/4, \infty\}} \leq \rho \leq \mathbf{1}_{\{1/2, \infty\}}$ . Then, combining inequality (1.33) of Kusuoka and Stroock (1984, Theorem 1.31) and Corollary 3.7 leads to

$$\sup_{|y-x| \geq r} p^N(0, x; s, y) \leq K(T) \frac{e^{-c \frac{r^2}{s}}}{s^{d/2}} \left(1 + \sqrt{\frac{s}{r^2}}\right),$$

where we use  $|\psi(X_s^N)|_{1,q} \leq K(T) e^{-c \frac{r^2}{s}} (1 + \sqrt{\frac{s}{r^2}})$ . This easily completes the proof in the case  $|x - y| \geq \sqrt{s}$ .  $\blacksquare$

## 4 Proof of Theorem 3

In the following,  $K(\cdot)$  denotes a generic non decreasing function (which may depend on  $d, b$  and  $\sigma$ ). To prove Theorem 3, we take advantage of Propositions 9 and 10. The scheme of the proof is the following

- Use a PDE and Itô's calculus to write the difference  $p^N(0, x; s, y) - p(0, x; s, y)$

$$\begin{aligned} &= \int_0^s \mathbb{E} \left[ \sum_{i=1}^d (b_i(\varphi(r), X_{\varphi(r)}^N) - b_i(r, X_r^N)) \partial_{x_i} p(r, X_r^N; s, y) \right. \\ &\quad \left. + \frac{1}{2} \sum_{i,j=1}^d (a_{ij}(\varphi(r), X_{\varphi(r)}^N) - a_{ij}(r, X_r^N)) \partial_{x_i x_j}^2 p(r, X_r^N; s, y) \right] dr := E_1 + E_2. \end{aligned} \quad (4.1)$$

- Prove the intermediate result  $\forall (r, x, y) \in [0, s] \times \mathbb{R}^d \times \mathbb{R}^d$  and  $c > 0$

$$\mathbb{E} \left[ \exp \left( -c \frac{|y - X_r^N|^2}{s - r} \right) \right] \leq K(T) \left( \frac{s - r}{s} \right)^{\frac{d}{2}} \exp \left( -c' \frac{|x - y|^2}{s} \right), \quad (4.2)$$

where  $c' > 0$ .



- Use Malliavin's calculus, Proposition 10 and the intermediate result, to show that each term  $E_1$  and  $E_2$  (see (4.1)) is bounded by  $\frac{K(T)T}{N} \frac{1}{s^{\frac{d+1}{2}}} \exp(-c \frac{|x-y|^2}{s})$ .

**Definition 11.** We say that a term  $E(x, s, y)$  satisfies property  $\mathcal{P}$  if  $\forall (x, s, y) \in \mathbb{R}^d \times ]0, T] \times \mathbb{R}^d$

$$|E(x, s, y)| \leq \frac{K(T)T}{N} \frac{1}{s^{\frac{d+1}{2}}} \exp\left(-c \frac{|x-y|^2}{s}\right). \quad (\mathcal{P})$$

#### 4.1 Proof of equality (4.1)

First, the transition density function  $(r, x) \mapsto p(r, x; s, y)$  satisfies the PDE

$$(\partial_r + \mathcal{L}_{(r,x)})p(r, x; s, y) = 0, \quad \forall r \in [0, s[, \forall x \in \mathbb{R}^d,$$

where  $\mathcal{L}_{(r,x)}$  is defined by  $\mathcal{L}_{(r,x)} = \sum_{i,j} a_{ij}(r, x) \partial_{x_i x_j}^2 + \sum_i b_i(r, x) \partial_{x_i}$ , and  $a_{ij}(r, x) = \frac{1}{2}[\sigma \sigma^*]_{ij}(r, x)$ . The function, as well as its first derivatives, are uniformly bounded by a constant depending on  $\epsilon$  for  $|s - r| \geq \epsilon$  (see Appendix A).

Second, since  $p^N(0, x; s, y)$  is a continuous function in  $s$  and  $y$  (convolution of Gaussian densities), we observe that

$$p^N(0, x; s, y) - p(0, x; s, y) = \lim_{\epsilon \rightarrow 0} \mathbb{E}[p(s - \epsilon, X_{s-\epsilon}^N; s, y) - p(0, x; s, y)].$$

Then, for any  $\epsilon > 0$ , Itô's formula leads to

$$\begin{aligned} \mathbb{E}[p(s - \epsilon, X_{s-\epsilon}^N; s, y) - p(0, x; s, y)] &= \mathbb{E}\left[\int_0^{s-\epsilon} \partial_r p(r, X_r^N; s, y) dr\right] \\ &+ \mathbb{E}\left[\int_0^{s-\epsilon} \sum_{i=1}^d b_i(\varphi(r), X_{\varphi(r)}^N) \partial_{x_i} p(r, X_r^N; s, y) dr + \frac{1}{2} \int_0^{s-\epsilon} \sum_{i,j=1}^d a_{ij}(\varphi(r), X_{\varphi(r)}^N) \partial_{x_i x_j}^2 p(r, X_r^N; s, y) dr\right]. \end{aligned}$$

From the PDE, the above equality becomes

$$\begin{aligned} \mathbb{E}[p(s - \epsilon, X_{s-\epsilon}^N; s, y) - p(0, x; s, y)] &= \mathbb{E}\left[\int_0^{s-\epsilon} \sum_{i=1}^d (b_i(\varphi(r), X_{\varphi(r)}^N) - b_i(r, X_r^N)) \partial_{x_i} p(r, X_r^N; s, y) dr\right] \\ &+ \frac{1}{2} \mathbb{E}\left[\int_0^{s-\epsilon} \sum_{i,j=1}^d (a_{ij}(\varphi(r), X_{\varphi(r)}^N) - a_{ij}(r, X_r^N)) \partial_{x_i x_j}^2 p(r, X_r^N; s, y) dr\right], \\ &:= \int_0^{s-\epsilon} \mathbb{E}[\phi(r)] dr, \end{aligned}$$

where  $\phi(r) = \sum_{i=1}^d (b_i(\varphi(r), X_{\varphi(r)}^N) - b_i(r, X_r^N)) \partial_{x_i} p(r, X_r^N; s, y) + \frac{1}{2} \sum_{i,j=1}^d (a_{ij}(\varphi(r), X_{\varphi(r)}^N) - a_{ij}(r, X_r^N)) \partial_{x_i x_j}^2 p(r, X_r^N; s, y)$ . To get (4.1), it remains to prove that  $\mathbb{E}(\phi(r))$  is integrable over  $[0, s]$ . We check it by looking at the rest of the proof.

## 4.2 Proof of the intermediate result (4.2)

We prove inequality (4.2).  $\mathbb{E}[\exp(-c\frac{|y-X_r^N|^2}{s-r})] = \int_{\mathbb{R}^d} \exp(-c\frac{|y-z|^2}{s-r})p^N(0, x; r, z)dz$ . Using Proposition 10, we get

$$\begin{aligned}\mathbb{E}\left[\exp\left(-c\frac{|y-X_r^N|^2}{s-r}\right)\right] &\leq \frac{K(T)}{r^{\frac{d}{2}}} \int_{\mathbb{R}^d} \exp\left(-c\frac{|y-z|^2}{s-r}\right) \exp\left(-c'\frac{|x-z|^2}{r}\right) dz \\ &\leq K(T)\Pi_{i=1}^d \int_{\mathbb{R}} \frac{1}{\sqrt{r}} \exp\left(-c\frac{|y_i-z_i|^2}{s-r}\right) \exp\left(-c'\frac{|x_i-z_i|^2}{r}\right) dz_i,\end{aligned}$$

and  $\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\frac{(s-r)}{2c}}} \exp(-c\frac{|y_i-z_i|^2}{s-r}) \frac{1}{\sqrt{2\pi\frac{r}{2c'}}} \exp(-c'\frac{|x_i-z_i|^2}{r}) dz_i$  is the convolution product of the density of two independant Gaussian random variables  $\mathcal{N}(-x_i, \frac{r}{2c'})$  and  $\mathcal{N}(y_i, \frac{s-r}{2c})$  computed at 0. Hence, the integral is equal to  $\frac{1}{\sqrt{2\pi(\frac{r}{2c'} + \frac{s-r}{2c})}} \exp\left(-\frac{|x_i-y_i|^2}{\frac{r}{c'} + \frac{s-r}{c}}\right)$ . Then,

$$\int_{\mathbb{R}} \frac{1}{\sqrt{r}} \exp\left(-c\frac{|y_i-z_i|^2}{s-r}\right) \exp\left(-c'\frac{|x_i-z_i|^2}{r}\right) dz_i \leq C \left(\frac{s-r}{s}\right)^{\frac{1}{2}} \exp\left(-c''\frac{|x_i-y_i|^2}{s}\right)$$

and (4.2) follows.

## 4.3 Upper bound for $E_1$

We recall that  $E_1 = \int_0^s \mathbb{E}\left[\sum_{i=1}^d (b_i(\varphi(r), X_{\varphi(r)}^N) - b_i(r, X_r^N)) \partial_{x_i} p(r, X_r^N; s, y)\right] dr$ . For each  $i$ , we apply Itô's formula to  $b_i(u, X_u^N)$  between  $u = \varphi(r)$  and  $u = r$ . We get

$$b_i(\varphi(r), X_{\varphi(r)}^N) - b_i(r, X_r^N) = \int_{\varphi(r)}^r \alpha_u^i du + \int_{\varphi(r)}^r \sum_{k=1}^q \beta_u^{i,k} dW_u^k, \quad (4.3)$$

where  $\alpha_u^i$  depends on  $\partial_t b, \partial_x b, \partial_x^2 b, \sigma$ , and  $\beta_u^i = -\nabla_x b_i(u, X_u^N) \sigma(\varphi(r), X_{\varphi(r)}^N)$ . Since  $b, \sigma$  belong to  $C_b^{1,3}$ ,  $\alpha^i$  and  $(\beta^{i,k})_{1 \leq k \leq q}$  are uniformly bounded. Using (4.3) and the duality formula (3.1) yield

$$\begin{aligned}E_1 &= \sum_{i=1}^d \int_0^s \left\{ \mathbb{E}\left[\int_{\varphi(r)}^r \partial_{x_i} p(r, X_r^N; s, y) \alpha_u^i du\right] + \mathbb{E}\left[\int_{\varphi(r)}^r \mathcal{D}_u(\partial_{x_i} p(r, X_r^N; s, y)) \cdot \beta_u^i du\right] \right\} dr \\ &:= E_{11} + E_{12},\end{aligned} \quad (4.4)$$

where  $\beta_u^i$  is a row vector of  $q$  components. We upper bound  $E_{11}$  and  $E_{12}$ .

**Bound for  $E_{11} = \sum_{i=1}^d \int_0^s \mathbb{E}[\int_{\varphi(r)}^r \partial_{x_i} p(r, X_r^N; s, y) \alpha_u^i du] dr$ .**

Since  $|\sum_{i=1}^d \partial_{x_i} p(r, X_r^N; s, y) \alpha_u^i| \leq |\alpha_u| |\partial_x p(r, X_r^N; s, y)|$  and  $\alpha_u$  is uniformly bounded in  $u$ , we have

$$|E_{11}| \leq C \frac{T}{N} \int_0^s \mathbb{E} |\partial_x p(r, X_r^N; s, y)| dr.$$

Besides that, from Proposition 13,  $|\partial_x p(r, X_r^N; s, y)| \leq \frac{K(T)}{(s-r)^{\frac{d+1}{2}}} \exp(-c \frac{|y-X_r^N|^2}{s-r})$ . Then,

$$|E_{11}| \leq K(T) \frac{T}{N} \int_0^s \frac{1}{(s-r)^{\frac{d+1}{2}}} \mathbb{E} \left[ \exp \left( -c \frac{|y-X_r^N|^2}{s-r} \right) \right] dr.$$

Using the intermediate result (4.2) yields

$$|E_{11}| \leq K(T) \frac{T}{N} \int_0^s \frac{1}{\sqrt{s-r}} \frac{1}{s^{\frac{d}{2}}} \exp \left( -c \frac{|x-y|^2}{s} \right) dr \leq K(T) \frac{T}{N} \frac{1}{s^{\frac{d-1}{2}}} \exp \left( -c \frac{|x-y|^2}{s} \right)$$

and thus,  $E_{11}$  satisfies property  $\mathcal{P}$  (see Definition 11).

**Bound for  $E_{12}$**   $= \sum_{i=1}^d \int_0^s \mathbb{E}[\int_{\varphi(r)}^r \mathcal{D}_u(\partial_{x_i} p(r, X_r^N; s, y)) \cdot \beta_u^i du] dr$ .

To rewrite  $E_{12}$ , we use the expression of  $\beta_u^i$  and Proposition 6, which gives  $\mathcal{D}_u(\partial_{x_i} p(r, X_r^N; s, y)) = \nabla_x(\partial_{x_i} p(r, X_r^N; s, y)) \sigma(\varphi(r), X_{\varphi(r)}^N)$ . Then,

$$E_{12} = - \int_0^s dr \int_{\varphi(r)}^r \sum_{i,k=1}^d \mathbb{E}[\partial_{x_i x_k}^2 p(r, X_r^N; s, y) [(\sigma \sigma^*)(\varphi(r), X_{\varphi(r)}^N) (\nabla_x b_i(u, X_u^N))^*]_k] du. \quad (4.5)$$

Using the integration by parts formula (3.2), we get that

$$E_{12} = - \int_0^s dr \int_{\varphi(r)}^r \sum_{i,k=1}^d \mathbb{E}[\partial_{x_i} p(r, X_r^N; s, y) H_{e_k}^i(r, u)] du$$

where  $e_k$  is a vector whose  $k$ -th component is 1 and other components are 0. From (3.3), we deduce  $\mathbb{E}[|H_{e_k}^i(r, u)|^p]^{1/p} \leq C \frac{K(T)}{r^{1/2}}$ , where  $C$  only depends on  $|\sigma|_\infty$ ,  $|\partial_x \sigma|_\infty$ ,  $|\partial_x b|_\infty$ ,  $|\partial_{xx}^2 b|_\infty$ . By the Hölder inequality, it follows that

$$|E_{12}| \leq K(T) \int_0^s dr \int_{\varphi(r)}^r \frac{1}{r^{1/2}} \mathbb{E}[|\partial_x p(r, X_r^N; s, y)|^{\frac{d+1}{d}}]^{\frac{d}{d+1}} du.$$

Using Proposition 13 leads to  $|\partial_x p(r, X_r^N; s, y)| \leq \frac{K(T)}{(s-r)^{\frac{d+1}{2}}} \exp(-c \frac{|y-X_r^N|^2}{s-r})$ , and combining this inequality with the intermediate result (4.2) yields

$$\mathbb{E}[|\partial_x p(r, X_r^N; s, y)|^{\frac{d+1}{d}}]^{d/(d+1)} \leq \frac{K(T)}{(s-r)^{\frac{d+1}{2}}} \left( \frac{s-r}{s} \right)^{\frac{d^2}{2(d+1)}} \exp \left( -c \frac{|y-x|^2}{s} \right). \quad (4.6)$$

Hence,  $E_{12}$  is bounded by

$$\frac{K(T)}{s^{\frac{d^2}{2(d+1)}}} \frac{T}{N} \exp \left( -c \frac{|y-x|^2}{s} \right) \int_0^s \frac{1}{r^{1/2}} \frac{1}{(s-r)^{\frac{d+1}{2} - \frac{d^2}{2(d+1)}}} dr.$$

The above integral equals  $s^{\frac{1}{2} - \frac{d+1}{2} + \frac{d^2}{2(d+1)}} B(\frac{1}{2}, \frac{1}{2(d+1)})$  where  $B$  is the function Beta. Thus  $|E_{12}| \leq \frac{K(T)}{s^{d/2}} \frac{T}{N} \exp(-c \frac{|y-x|^2}{s})$ , and  $E_{12}$  satisfies property  $\mathcal{P}$ .

#### 4.4 Upper bound for $E_2$

We recall  $E_2 = \frac{1}{2} \int_0^s \mathbb{E}[\sum_{i,j=1}^d (a_{ij}(\varphi(r), X_{\varphi(r)}^N) - a_{ij}(r, X_r^N)) \partial_{x_i x_j}^2 p(r, X_r^N; s, y)] dr$ . As we did for  $E_1$ , we apply Itô's formula to  $a_{ij}(u, X_u^N)$  between  $\varphi(r)$  and  $r$ . We get  $a_{ij}(\varphi(r), X_{\varphi(r)}^N) - a_{ij}(r, X_r^N) = \int_{\varphi(r)}^r \gamma_u^{ij} du + \int_{\varphi(r)}^r \delta_u^{ij} dW_u$ , where  $\gamma_u^{ij}$  depends on  $\sigma, \partial_t \sigma, \partial_x \sigma, b, \partial_{xx}^2 \sigma$  and  $\delta_u^{ij}$  is a row vector of size  $q$ , with  $l$ -th component  $(\delta_u^{ij})_l = -\sum_{k=1}^d \partial_{x_k} a_{ij}(u, X_u^N) \sigma_{kl}(\varphi(r), X_{\varphi(r)}^N)$ . Then, the duality formula (3.1) leads to

$$\begin{aligned} E_2 &= \sum_{i,j=1}^d \int_0^s \{ \mathbb{E}[\int_{\varphi(r)}^r \partial_{x_i x_j}^2 p(r, X_r^N; s, y) \gamma_u^{ij} du + \mathbb{E}[\int_{\varphi(r)}^r \mathcal{D}_u(\partial_{x_i x_j}^2 p(r, X_r^N; s, y)) \cdot \delta_u^{ij} du] \} dr \\ &:= E_{21} + E_{22}. \end{aligned}$$

**Bound for  $E_{21}$**   $= \sum_{i,j=1}^d \int_0^s \mathbb{E}[\int_{\varphi(r)}^r \partial_{x_i x_j}^2 p(r, X_r^N; s, y) \gamma_u^{ij} du] dr$ .

As  $\sigma, b, \partial_t \sigma, \partial_x \sigma, \partial_{xx}^2 \sigma$  are  $C_b^1$  in space,  $\gamma_u^{ij}$  has the same smoothness properties as the term  $[(\sigma \sigma^*)(\varphi(r), X_{\varphi(r)}^N)(\nabla_x b_i(u, X_u^N))^*]_k$  appearing in (4.5). Thus,  $E_{21}$  can be treated as  $E_{12}$  and satisfies to the same estimate.

**Bound for  $E_{22}$**   $= \sum_{i,j=1}^d \int_0^s \mathbb{E}[\int_{\varphi(r)}^r \mathcal{D}_u(\partial_{x_i x_j}^2 p(r, X_r^N; s, y)) \cdot \delta_u^{ij} du] dr$ .

To rewrite  $E_{22}$ , we use the expression of  $\delta_u^{ij}$  and Proposition 6, which asserts  $\mathcal{D}_u(\partial_{x_i x_j}^2 p(r, X_r^N; s, y)) = \nabla_x(\partial_{x_i x_j}^2 p(r, X_r^N; s, y)) \sigma(\varphi(r), X_{\varphi(r)}^N)$ . Thus,

$$E_{22} = - \sum_{i,j,k=1}^d \int_0^s dr \int_{\varphi(r)}^r \mathbb{E}[\partial_{x_i x_j x_k}^3 p(r, X_r^N; s, y) [(\sigma \sigma^*)(\varphi(r), X_{\varphi(r)}^N)(\nabla_x a_{ij}(u, X_u^N))^*]_k] du.$$

To complete this proof, we split  $E_{22}$  in two terms :  $E_{22}^1$  (resp  $E_{22}^2$ ) corresponds to the integral in  $r$  from 0 to  $\frac{s}{2}$  (resp. from  $\frac{s}{2}$  to  $s$ ).

- On  $[0, \frac{s}{2}]$ ,  $E_{22}^1$  is bounded by  $C \frac{T}{N} \int_0^{\frac{s}{2}} \mathbb{E}[|\partial_{x_i x_j x_k}^3 p(r, X_r^N; s, y)|] dr$ . Using Proposition 13 and (4.2), it gives

$$|E_{22}^1| \leq \frac{K(T)T}{N} \frac{1}{s^{d/2}} \exp\left(-c \frac{|x-y|^2}{s}\right) \int_0^{\frac{s}{2}} \frac{1}{(s-r)^{3/2}} dr.$$

Hence,  $E_{22}$  satisfies  $\mathcal{P}$ .

- On  $[\frac{s}{2}, s]$ , we use the integration by parts formula (3.2) of Proposition 9, with  $|\alpha| = 2$ .

$$E_{22}^2 = - \sum_{i,j,k=1}^d \int_{\frac{s}{2}}^s dr \int_{\varphi(r)}^r \mathbb{E}[\partial_{x_i} p(r, X_r^N; s, y) H_{e_{jk}}^i] du,$$

where  $e_{jk}$  is a vector full of zeros except the  $j$ -th and the  $k$ -th components. Using Hölder's inequality and (3.3) (remember that  $\sigma \in C_b^{1,3}$ ), we obtain

$$|E_{22}^2| \leq K(T) \frac{T}{N} \int_{\frac{s}{2}}^s \frac{1}{r} \mathbb{E}[|\partial_x p(r, X_r^N; s, y)|^{\frac{d+1}{d}}]^{\frac{d}{d+1}} dr. \quad (4.7)$$

By applying (4.6), we get

$$|E_{22}^2| \leq K(T) \frac{T}{N} \frac{1}{s^{1+\frac{d^2}{2(d+1)}}} \exp\left(-c \frac{|x-y|^2}{s}\right) \int_{\frac{s}{2}}^s \frac{1}{(s-r)^{\frac{2d+1}{2d+2}}} dr,$$

and the result follows.

## A Bounds for the transition density function and its derivatives

We bring together classical results related to bounds for the transition probability density of  $X$  defined by (1.1).

**Proposition 12** (Aronson (1967)). *Assume that the coefficients  $\sigma$  and  $b$  are bounded measurable functions and that  $\sigma$  is uniformly elliptic. There exist positive constants  $K, \alpha_0, \alpha_1$  s.t. for any  $x, y$  in  $\mathbb{R}^d$  and any  $0 \leq t < s \leq T$ , one has*

$$\frac{K^{-1}}{(2\pi\alpha_1(s-t))^{\frac{d}{2}}} e^{-\frac{|x-y|^2}{2\alpha_1(s-t)}} \leq p(t, x; s, y) \leq K \frac{1}{(2\pi\alpha_2(s-t))^{\frac{d}{2}}} e^{-\frac{|x-y|^2}{2\alpha_2(s-t)}}. \quad (\text{A.1})$$

**Proposition 13** (Friedman (1964)). *Assume that the coefficients  $b$  and  $\sigma$  are Hölder continuous in time,  $C_b^2$  in space and that  $\sigma$  is uniformly elliptic. Then,  $\partial_x^{m+a} \partial_y^b p(t, x; s, y)$  exist and are continuous functions for all  $0 \leq |a| + |b| \leq 2, |m| = 0, 1$ . Moreover, there exist two positive constants  $c$  and  $K$  s.t. for any  $x, y$  in  $\mathbb{R}^d$  and any  $0 \leq t < s \leq T$ , one has*

$$|\partial_x^{m+a} \partial_y^b p(t, x; s, y)| \leq \frac{K}{(s-t)^{(|m|+|a|+|b|+d)/2}} \exp\left(-c \frac{|y-x|^2}{s-t}\right).$$

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